XX. TRUSS ELEMENT

XX.1 INTRODUTION

Plane truss structure is a stable structure on the basis of a triangle, as shown in Fig.XX.1.1. The end of a member is pin junction which does not transmit a moment. As for the truss members that constitute plane truss structure, only axial forces act, bending moments and shear forces do not apply. In addition, middle loads will not act on the truss members.

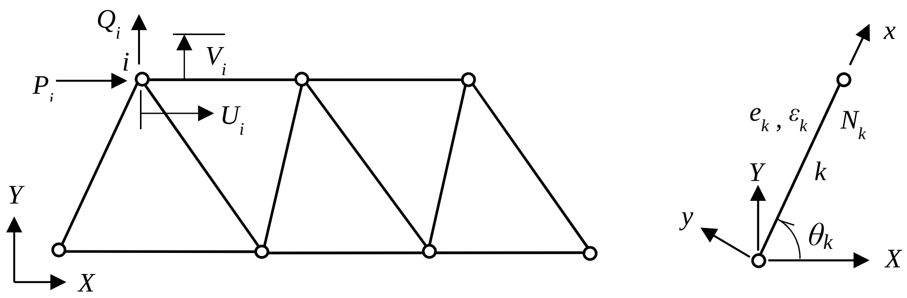


Figure XX.1.1 Plane truss structure

The characteristics of the truss element can be summarized as follows:

1) A truss element is a slender member (length is much larger than the cross-section).
2) It is a two-force member *i.e.* it can only support an axial load and cannot support a bending load. Members are joined by pins (no translation at the constrained node, but free to rotate in any direction).
3) The cross-sectional dimensions and elastic properties of each member are constant along its length.
4) The element may interconnect in a 2-D or 3-D configuration in space.

The element is mechanically equivalent to a spring, since it has no stiffness against applied loads except those acting along the axis of the member

XX.2 LOCAL AND GLOBAL COODINATES

XX.2.1 Relationship between Local and Global Displacements

We start by looking at the truss element shown in Figure 2.1. This element attaches to two nodes, *i* and *j*. In the figure we are showing two coordinate systems. One is a one dimensional coordinate system that aligns with the length of the element. We will call this the local coordinate system. The other is a two dimensional coordinate system that does not align with the element. We will call this the global coordinate system. The [x, y] coordinates are the local coordinates for the element and [X, Y] are the global coordinates.

We can convert the displacements shown in the local coordinate system by looking at the following figure. u_{ik}, v_{ik}, u_{jk} and v_{jk} represent displacements in the local coordinate system and U_{ik}, V_{ik}, U_{jk} and V_{jk} represent displacements in the [X, Y] (global) coordinate system.

Local coordinate system [x, y]

Nodal displacement of member k in the x direction : u_{ik}, u_{jk}

Nodal displacement of member k in the y direction : v_{ik}, v_{jk}

Nodal force of member k in the x direction : p_{ik}, p_{jk}

Nodal force of member k in the y direction : q_{ik}, q_{jk}

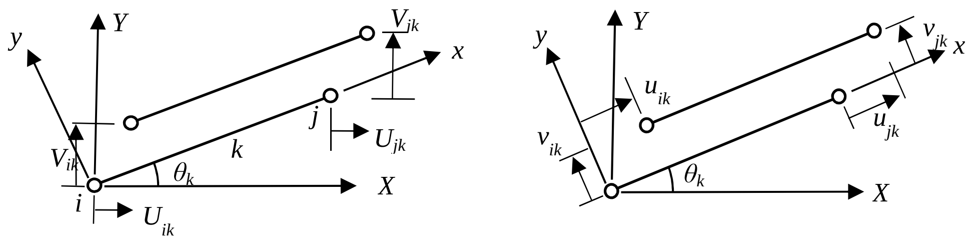
Global coordinate system [X, Y]

Nodal displacement of member k in the X direction : U_{ik}, U_{jk}

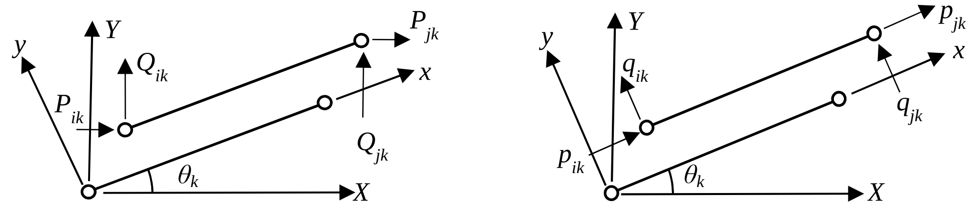
Nodal displacement of member k in the Y direction : V_{ik}, V_{jk}

Nodal force of member k in the X direction : P_{ik}, P_{jk}

Nodal force of member k in the Y direction : Q_{ik}, Q_{jk}



(a) Nodal displacements of Local coordinate system (b) Nodal displacements of Local coordinate system



(a) Nodal forces of Local coordinate system (b) Nodal forces of Local coordinate system

Figure XX.2.1.1 Nodal displacements and nodal forces of Local and Global coordinate systems

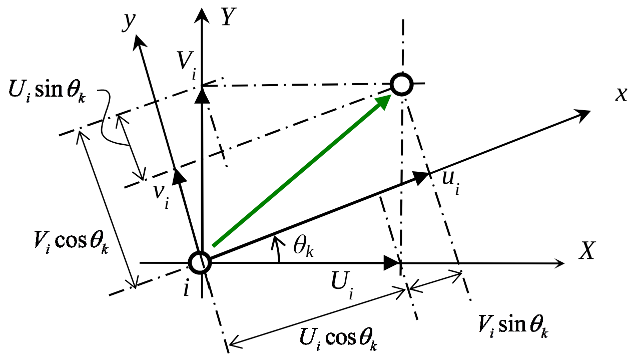


Figure XX.2.1.2 Nodal displacements and nodal forces of Local and Global coordinate systems

Let us consider the relation between the nodal displacements U_{ik}, V_{ik} in the global coordinate system and the nodal displacements u_{ik}, v_{ik} in the local coordinate system. From a geometric relation as shown in Fig.XX.2.1.2, the governing equations relating the two coordinate values (U_{ik}, V_{ik}, u_{ik}, and v_{ik}) are given as

$$\begin{aligned} u_{ik} &= \ \ U_{ik}\cos\theta_k + V_{ik}\sin\theta_k \\ v_{ik} &= -U_{ik}\sin\theta_k + V_{ik}\cos\theta_k \end{aligned} \tag{XX.2.1.1}$$

Similarly, for the nodal displacements of Node j, we can obtain the following relations.

$$\begin{aligned} u_{jk} &= \ \ U_{jk}\cos\theta_k + V_{jk}\sin\theta_k \\ v_{jk} &= -U_{jk}\sin\theta_k + V_{jk}\cos\theta_k \end{aligned} \tag{XX.2.1.2}$$

In matrix form

$$\begin{Bmatrix} u_{ik} \\ v_{ik} \\ u_{jk} \\ v_{jk} \end{Bmatrix} = \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix} \tag{XX.2.1.3}$$

Similarly, for the nodal forces, we also can obtain the following relations.

$$\begin{aligned} p_{ik} &= \ \ P_{ik}\cos\theta_k + Q_{ik}\sin\theta_k \\ q_{ik} &= -P_{ik}\sin\theta_k + Q_{ik}\cos\theta_k \end{aligned} \tag{XX.2.1.4}$$

$$\begin{aligned} p_{ik} &= \ \ P_{ik}\cos\theta_k + Q_{ik}\sin\theta_k \\ q_{ik} &= -P_{ik}\sin\theta_k + Q_{ik}\cos\theta_k \end{aligned} \tag{XX.2.1.5}$$

In matrix form

$$\begin{Bmatrix} p_{ik} \\ q_{ik} \\ p_{jk} \\ q_{jk} \end{Bmatrix} = \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{Bmatrix} P_{ik} \\ Q_{ik} \\ P_{jk} \\ Q_{jk} \end{Bmatrix} \tag{XX.2.1.6}$$

Accordingly, the relationship between the values in the global coordinate and the values in the local coordinate is expressed in the matrix form.

$$\{p_k\} = [T_k]\{P_k\} \tag{XX.2.1.7}$$

$$\{d_k\} = [T_k]\{D_k\} \tag{XX.2.1.8}$$

$$\{P_k\} = \begin{Bmatrix} P_{ik} \\ Q_{ik} \\ P_{jk} \\ Q_{jk} \end{Bmatrix}, \quad \{p_k\} = \begin{Bmatrix} p_{ik} \\ q_{ik} \\ p_{jk} \\ q_{jk} \end{Bmatrix}, \quad \{D_k\} = \begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix}, \quad \{d_k\} = \begin{Bmatrix} u_{ik} \\ v_{ik} \\ u_{jk} \\ v_{jk} \end{Bmatrix} \tag{XX.2.1.9}$$

$$[T_k] = \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & \sin\theta_k & \cos\theta_k \end{bmatrix} \tag{XX.2.1.10}$$

in which $[T_k]$ is the transformation matrix. $\{D_k\}$, $\{P_k\}$ are nodal displacement and force vector in the global coordinate, and $\{d_k\}$, $\{p_k\}$ are nodal displacement and force vector in the local coordinate.

XX.3 FINEIT ELMENT EQUATION IN LOCAL COODINATE SYSTEM

Now, we will derive the finite element equation in local coordinate system. Consider the truss element shown in Fig.XX.3.1.1, with nodes i and j, displacements v_i, u_i, u_j and v_j, and forces p_{ik}, q_{ik}, p_{jk} and q_{jk}.

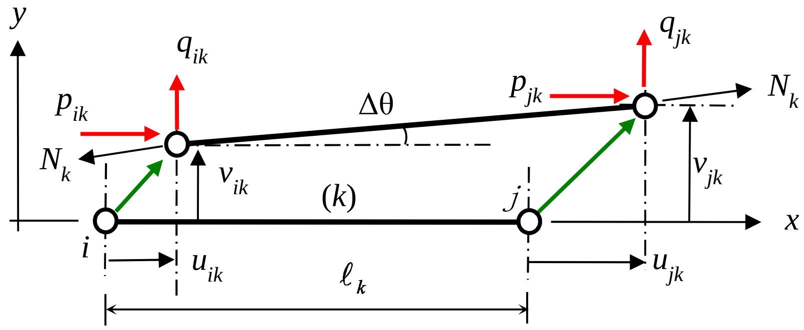


Figure XX.3.1.1

XX.3.1 Strain-displacement relation:

The strain-displacement relation in the truss element is calculated by interpolation of deflection values shared by nodes of the element. The relationship strain ε_k and deflection e_k is given by

$$\varepsilon_k = \frac{\ell_k' - \ell_k}{\ell_k} = \frac{(\ell_k + e_k) - \ell_k}{\ell_k} = \frac{e_k}{\ell_k} \tag{XX.3.1.1}$$

in which ℓ_k is a member length before deformation, and ℓ_k' is a member length after deformation, and e_k is a elongation of the truss element. The strain ε_k is expressed by nodal displacements and the member length.

$$\varepsilon_k = \frac{\sqrt{(\ell_k + u_{jk} - u_{ik})^2 + (v_{jk} - v_{ik})^2} - \ell_k}{\ell_k} = \sqrt{\left(1 + \frac{u_{jk} - u_{ik}}{\ell_k}\right)^2 + \left(\frac{v_{jk} - v_{ik}}{\ell_k}\right)^2} - 1$$

$$= \sqrt{1 + 2\frac{u_{jk} - u_{ik}}{\ell_k} + \left(\frac{u_{jk} - u_{ik}}{\ell_k}\right)^2 + \left(\frac{v_{jk} - v_{ik}}{\ell_k}\right)^2} - 1 \qquad \text{(XX.3.1.2)}$$

When the terms, $\frac{u_{jk} - u_{ik}}{\ell_k}$ and $\frac{v_{jk} - v_{ik}}{\ell_k}$ in above the equation are small enough, the above equation can be approximated as

$$\varepsilon_k \approx \sqrt{1 + 2\frac{u_{jk} - u_{ik}}{\ell_k}} - 1 \cong 1 + \frac{u_{jk} - u_{ik}}{\ell_k} - 1 = \frac{u_{jk} - u_{ik}}{\ell_k} \qquad \text{(XX.3.1.3)}$$

XX.3.2 Stress strain relation (constitutive equation)

The stress strain relation of the truss element is adopt the hook's law,

$$\sigma_k = E_k \varepsilon_k \qquad \text{(XX.3.2.1)}$$

where, E_k is a young's modulus or modulus of elasticity.

XX.3 Finite element equation in Local Coordinate System

From the above relationship, an axial force, N_k, of the truss element is given by

$$N_k = E_k A_k \varepsilon_k = \frac{E_k A_k}{\ell_k} e_k = \frac{E_k A_k}{\ell_k}(u_j - u_i) = k_k (u_j - u_i) \qquad \text{(XX.3.3.1)}$$

where A_k is a cross-section area of the truss element, and k_k is an axial stiffness of the truss element.

$$k_k = \frac{E_k A_k}{\ell_k} \qquad \text{(XX.3.3.2)}$$

As illustrated in Fig.XX.3.1.1, the applied forces p_{ik}, q_{ik}, p_{jk} and q_{jk} are expressed by the axial force, and the finite element equation in local coordinates is given as,

$$\begin{aligned}
p_{ik} &= -N_k \cos\Delta\theta \approx -N_k = -k_k (u_{jk} - u_{ik}) = k_k u_{ik} - k_k u_{jk} \\
q_{ik} &= -N_k \sin\Delta\theta \approx 0 \\
p_{jk} &= N_k \cos\Delta\theta \approx N_k = k_k (u_{jk} - u_{ik}) = -k_k u_{ik} + k_k u_{jk} \\
q_{jk} &= N_k \sin\Delta\theta \approx 0
\end{aligned} \qquad \text{(XX.3.3.3)}$$

in which the following equations are assumed in infinitesimal deformation theory.

$$\cos\Delta\theta \approx 1 , \quad \sin\Delta\theta \approx 0$$

As a matrix form of Eq.(XX.3.3.3),

$$\{p_k\} = [k_k]\{d_k\} \qquad \text{(XX.3.3.4)}$$

$$[k_k] = k_k \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad k_k = \frac{E_k \cdot A_k}{\ell_k} \qquad \text{(XX.3.3.5)}$$

Here, Eq.(XX.3.3.4) is a finite element equation in local coordinate system, [k_k] is a stiffness matrix of a truss element in local coordinates. This equation will be converted to global coordinate system, which can be used to generate a global structural equation for the given structure.

XX.4 FINEIT ELMENT EQUATION IN GLOBAL COODINATE SYSTEM

Using the relationships between local and global deflections and forces, we can convert an element equation from a local coordinate system to a global system. Substituting Eqs.(XX.2.1.7) and (XX.2.1.8) into the element local equilibrium, Eq.(XX.3.3.4), we obtain

$$[T_k]\{P_k\} = [k_k][T_k]\{D_k\} \tag{XX.4.1.1}$$

Multiplying both sides of this matrix equation by the inverse of the transformation matrix, that is, by $[T_k]^{-1}$, we obtain

$$\{P_k\} = [T_k]^{-1}[k_k][T_k]\{D_k\} \tag{XX.4.1.2}$$

For this and most cases, the transformation matrix $[T_k]^T$ has the property of orthogonality. Therefore, the inverse is the same sa the transpose $[T_k]^T$.

$$[T_k]^{-1} = [T_k]^T \tag{XX.4.1.3}$$

Finite element equation of the truss element in global coordinates is expressed as

$$\{P_k\} = [K_k]\{D_k\} \tag{XX.4.1.4}$$

in which $[\,K_k\,]$ is a stiffness matrix of a truss element in global coordinates. The stiffness matrix is expressed as

$$[K_k] = [T_k]^T[k_k][T_k]$$

$$= \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & -\sin\theta_k & \cos\theta_k \end{bmatrix}^T \frac{E_k \cdot A_k}{\ell_k} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & -\sin\theta_k & \cos\theta_k \end{bmatrix}$$

$$= \frac{E_k \cdot A_k}{\ell_k} \begin{bmatrix} \cos^2\theta_k & \sin\theta_k\cos\theta_k & -\cos^2\theta_k & -\sin\theta_k\cos\theta_k \\ \sin\theta_k\cos\theta_k & \sin^2\theta_k & -\sin\theta_k\cos\theta_k & -\sin^2\theta_k \\ -\cos^2\theta_k & -\sin\theta_k\cos\theta_k & \cos^2\theta_k & \sin\theta_k\cos\theta_k \\ -\sin\theta_k\cos\theta_k & -\sin^2\theta_k & \sin\theta_k\cos\theta_k & \sin^2\theta_k \end{bmatrix} \tag{XX.4.1.5}$$

Each of its terms has a physical significance, representing the contribution of one of the displacements to one of the forces. The global system of equations is formed by combining the element stiffness matrices from each truss element in turn, so their computation is central to the method of matrix structural analysis. This matrix has several noteworthy characteristics:

- The matrix is symmetric
- Since there are 4 unknown displacements (DOFs), the matrix size is a 4 x 4.
- The matrix represents the stiffness of a single element.

XX.5 ASSEMBLING THE STIFFNESS MATRICES

The next step is to consider an assemblage of many truss elements connected by pin joints. Each element meeting at a joint, or node, will contribute an external force. To maintain static equilibrium, all element force contributions $\{P_k\}$at a given node must sum to the force that is externally applied at that node:

$$\{P\} = \sum_{elem}\{P_k\} = \sum_{elem}[K_k]\{D_k\} = [K]\{D\} \tag{XX.5.1.1}$$

Here, $\{ P \}$ is a external force, $\{ D \}$ is a displacement vector, and $[K]$ is a overall, or "global" stiffness matrix. Each element stiffness matrix $[K_k]$ is added to the appropriate location of the overall stiffness matrix $[K]$ that relates all of the truss displacements and forces. This process is called "*assembly*".

Eq.(XX.5.1.1) is a simultaneous linear equations. Solving the Eq.(XX.5.1.1), the displacement vector can be calculated as

$$\{D\} = [K]^{-1}\{P\} \tag{XX.5.1.2}$$

XX.6 CALCULATION OF THE AXIAL FORCES OF TRUSS ELEMENT

To evaluate an axial force of each member, the nodal displacements of each member are calculated based on the result of displacement vector in Eq.(XX.5.12). The elongation e_k of truss member k is expressed with the nodal displacements of the global coordinate system.

$$\begin{aligned} e_k &= u_{jk} - u_{ik} \\ &= \left(U_{jk}\cos\theta_k + V_{jk}\sin\theta_k\right) - \left(U_{ik}\cos\theta_k + V_{ik}\sin\theta_k\right) \\ &= -\cos\theta_k\, U_{ik} - \sin\theta_k\, V_{ik} + \cos\theta_k\, U_{jk} + \sin\theta_k\, V_{jk} \\ &= \left[-\cos\theta_k \quad -\sin\theta_k \quad \sin\theta_k \quad \cos\theta_k\right]\begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix} \end{aligned} \tag{XX.6.1.1}$$

By using the nodal displacements, the axial force N_k of truss member k is expressed as

$$N_k = \frac{E_k \cdot A_k}{\ell_k} e_k = \frac{E_k \cdot A_k}{\ell_k}\left[-\cos\theta_k \quad -\sin\theta_k \quad \sin\theta_k \quad \cos\theta_k\right]\begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix} \tag{XX.6.1.2}$$

In general, substituting Eq.(XX.2.1.8) into Eq.(XX.3.3.4), the nodal force vector $\{p_k\}$ in the local coordinate is expressed as

$$\{p_k\} = [k_k]\{d_k\} = [k_k][T_k]\{D_k\} \tag{XX.3.3.4}$$

$$\begin{Bmatrix} p_{ik} \\ q_{ik} \\ p_{jk} \\ q_{jk} \end{Bmatrix} = \frac{E_k \cdot A_k}{\ell_k}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 & 0 \\ -\sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & \cos\theta_k & \sin\theta_k \\ 0 & 0 & -\sin\theta_k & \cos\theta_k \end{bmatrix}\begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix}$$

$$= \frac{E_k \cdot A_k}{\ell_k} \begin{bmatrix} \cos\theta_k & \sin\theta_k & -\cos\theta_k & -\sin\theta_k \\ 0 & 0 & 0 & 0 \\ -\cos\theta_k & -\sin\theta_k & \cos\theta_k & \sin\theta_k \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_{ik} \\ V_{ik} \\ U_{jk} \\ V_{jk} \end{Bmatrix} \qquad \text{(XX.3.3.4)}$$

in which the value of third row, p_{jk}, of the vector $\{p_k\}$ corresponds to the axial force N_k.

XX.7 COMPUTAIONAL PROCEDURE

XX.7.1 Geometry of the structure

There are 3 nodes and 3 elements making up the truss structure. We are going to do a two dimensional analysis so each node is constrained to move in only the X or Y direction. We call these directions of motion degrees of freedom or *dof* for short. There are 3 nodes and 3 degrees of freedom (two degrees of freedom for each node). Nodal displacements are assumed to be D_1, D_2 and D_3 as shown in Fig.XX.7.1.1. Applied external forces are assumed to be P_1, P_2 and P_3 corresponding to D_1, D_2 and D_3, respectively.

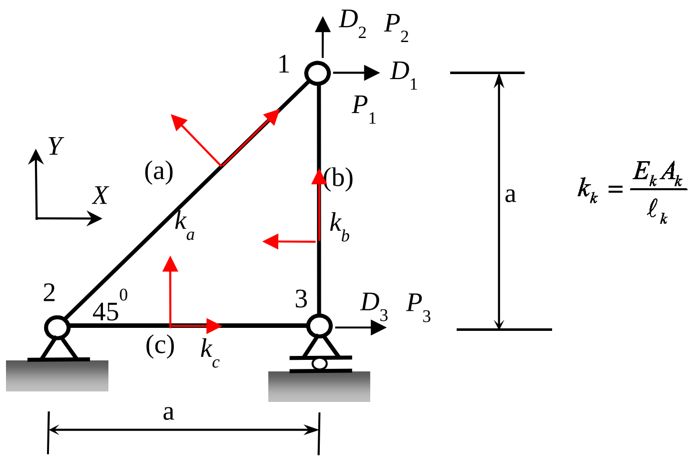


Fig.XX.7.1.1

We can locate each node by its coordinates. Table XX.7.1 shows the coordinates of the nodes in the problem we are solving. We can use these coordinates to determine the lengths and angles of the elements.

Table XX.7.1.1 Coordinates of the nodes in the truss.

Node	X	Y
1	a	a
2	0	0
3	a	0

Table XX.7.1.2 Elements and Nodes they connect in the truss.

Element	From Node	To Node	Length	Cosine	Sine
(a)	2	1	$\sqrt{2}$ a	$1/\sqrt{2}$	$1/\sqrt{2}$
(b)	2	3	a	0	1
(c)	3	1	a	1	0

Each element can be described as extending from one node to another. This also can be defined in Table XX.7.1.2. From these two tables we can derive the lengths of each element and the cosine and sine of their orientation. This is shown in the table below.

Axial stiffness are assumed to be expressed as k_k

$$k_k = \frac{E_k \cdot A_k}{\ell_k} \quad ; \quad k = a, b \text{ and } c \tag{XX.7.1.1}$$

in which E_k, A_k and ℓ_k are the young's modulus, the cross-section area and length of the member.

XX.7.2 Find the stiffness matrix for each element

In the previous sections we developed the stiffness matrix for an element.

(1) Element (a)

$$\theta_a = 45 \text{ deg.}, \quad \sin\theta_a = \frac{1}{\sqrt{2}}, \ \cos\theta_a = \frac{1}{\sqrt{2}}, \ \sin^2\theta_a = \frac{1}{2}, \ \cos^2\theta_a = \frac{1}{2}, \ \sin\theta_a\cos\theta_a = \frac{1}{2}$$

$$k_a = \frac{E_a A_a}{\ell_a} = \frac{E\,A}{\sqrt{2}\,a}$$

$$\begin{Bmatrix} P_{2a} \\ Q_{2a} \\ P_{1a} \\ Q_{1a} \end{Bmatrix} = \frac{k_a}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} U_{2a} \\ V_{2a} \\ U_{1a} \\ V_{1a} \end{Bmatrix} = \frac{k_a}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} \qquad \text{(XX.7.2.1)}$$

(2) Element (b)

$$\theta_b = 90 \text{ deg.}, \quad \sin\theta_b = 1, \ \cos\theta_b = 0, \ \sin^2\theta_b = 1, \ \cos^2\theta_b = 0, \ \sin\theta_b\cos\theta_b = 0$$

$$k_b = \frac{E_b A_b}{\ell_b} = \frac{E\,A}{a}$$

$$\begin{Bmatrix} P_{3b} \\ Q_{3b} \\ P_{1b} \\ Q_{1b} \end{Bmatrix} = k_b\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} U_{3b} \\ V_{3b} \\ U_{1b} \\ V_{1b} \end{Bmatrix} = k_b\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} D_3 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} \qquad \text{(XX.7.2.2)}$$

(3) Element (c)

$$\theta_c = 0 \text{ deg.}, \quad \sin\theta_c = 0, \ \cos\theta_c = 1, \ \sin^2\theta_c = 0, \ \cos^2\theta_c = 1, \ \sin\theta_c\cos\theta_c = 0$$

$$k_b = \frac{E_b A_b}{\ell_b} = \frac{E\,A}{a}$$

$$\begin{Bmatrix} P_{2c} \\ Q_{2c} \\ P_{3c} \\ Q_{3c} \end{Bmatrix} = k_c\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_{2c} \\ V_{2c} \\ U_{3c} \\ V_{3c} \end{Bmatrix} = k_c\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ D_3 \\ 0 \end{Bmatrix} \qquad \text{(XX.7.2.3)}$$

XX.7.3 Assembling the global stiffness matrices

Since there are 3 displacements (or DOFs), D_1 through D_3, the matrix is 3 x 3. Now, we will place the individual matrix element from the element stiffness matrices into the global matrix according to their position of row and column members.

(1) Element (a)

The nodal forces, P_{1a} and Q_{1a} corresponding to D_1 and D_2 directions are expressed as

$$P_{1a} = \frac{k_a}{2}D_1 + \frac{k_a}{2}D_2 \qquad \text{(XX.7.3.1)}$$

$$Q_{1a} = \frac{k_a}{2} D_1 + \frac{k_a}{2} D_2 \tag{XX.7.3.2}$$

In the matrix form,

$$\begin{Bmatrix} P_{1a} \\ Q_{1a} \\ 0 \end{Bmatrix} = \begin{bmatrix} k_a/2 & k_a/2 & 0 \\ k_a/2 & k_a/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \tag{XX.7.3.3}$$

(2) Element (b)

The nodal forces, P_{3b}, P_{1b}, and Q_{1a} corresponding to D_3, D_1 and D_2 directions are expressed as

$$P_{3b} = 0 \tag{XX.7.3.4}$$

$$P_{1b} = 0 \tag{XX.7.3.5}$$

$$Q_{1b} = k_b D_2 \tag{XX.7.3.6}$$

In the matrix form,

$$\begin{Bmatrix} P_{1b} \\ Q_{1b} \\ P_{3b} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_b & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \tag{XX.7.3.7}$$

(3) Element (c)

The nodal force P_{3c} corresponding to D_3 direction is expressed as

$$P_{3c} = k_c D_3 \tag{XX.7.3.8}$$

In the matrix form,

$$\begin{Bmatrix} 0 \\ 0 \\ P_{3c} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \tag{XX.7.3.9}$$

Assembling all the terms for elements (a), (b) and (c), we get the complete matrix equation of the structure.

$$\begin{aligned} P_1 &= P_{1a} + P_{1b} &&= \frac{k_a}{2} D_1 + \frac{k_a}{2} D_2 + 0 = \frac{k_a}{2} D_1 + \frac{k_a}{2} D_2 \\ P_2 &= Q_{1a} + Q_{1b} &&= \frac{k_a}{2} D_1 + \frac{k_a}{2} D_2 + k_b D_2 = \frac{k_a}{2} D_1 + \left(\frac{k_a}{2} + k_b \right) D_2 \\ P_3 &= P_{3c} &&= k_c D_3 \end{aligned} \tag{XX.7.3.10}$$

In the matrix form,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_a/2 & k_a/2 & \mathrm{O} \\ k_a/2 & k_a/2 + k_b & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & k_c \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \tag{XX.7.3.11}$$

In other word, adding the Eqs.(XX.7.3.3), (XX.7.3.7) and (XX.7.3.9), we get

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_{1a} \\ Q_{1a} \\ 0 \end{Bmatrix} + \begin{Bmatrix} P_{1b} \\ Q_{1b} \\ P_{3b} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P_{3c} \end{Bmatrix} = \begin{bmatrix} k_a/2 & k_a/2 & 0 \\ k_a/2 & k_a/2 + k_b & 0 \\ 0 & 0 & k_c \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \qquad \text{(XX.7.3.12)}$$

XX.7.4 Solving the matrix equation

Writing the matrix equation into algebraic linear equations, we get

$$k_c D_3 = P_3 \qquad \text{(XX.7.4.1)}$$

$$\begin{bmatrix} k_a/2 & k_a/2 \\ k_a/2 & k_a/2 + k_b \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \qquad \text{(XX.7.4.2)}$$

Solving Eqs.(XX.7.4.1) and (XX.7.4.2), we get

$$D_3 = P_3 / k_c \qquad \text{(XX.7.4.3)}$$

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{2}{k_a k_b} \begin{bmatrix} \frac{k_a}{2} + k_b & -\frac{k_a}{2} \\ -\frac{k_a}{2} & \frac{k_a}{2} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_b} + \frac{2}{k_a} & -\frac{1}{k_b} \\ -\frac{1}{k_b} & \frac{1}{k_b} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} \left(\frac{1}{k_b} + \frac{2}{k_a}\right) P_1 - \frac{1}{k_b} P_2 \\ -\frac{1}{k_b} P_1 + \frac{1}{k_b} P_2 \end{Bmatrix} \qquad \text{(XX.7.4.4)}$$

Accordingly, the displacements of the structure are

$$D_1 = \left(\frac{1}{k_b} + \frac{2}{k_a}\right) P_1 - \frac{1}{k_b} P_2 \,, \;\; D_2 = -\frac{1}{k_b} P_1 + \frac{1}{k_b} P_2 \,, \;\; D_3 = \frac{P_3}{k_3} \qquad \text{(XX.7.4.5)}$$

XX.7.5 Axial force of each element

The axial force of each truss element is calculated by Eq.(XX.6.1.2).

(1) Element (a)

$$N_a = k_a \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} = \frac{k_a}{\sqrt{2}} (D_1 + D_2)$$

$$= \frac{k_a}{\sqrt{2}} \left(\left(\frac{1}{k_b} + \frac{2}{k_a}\right) P_1 - \frac{1}{k_b} P_2 - \frac{1}{k_b} P_1 + \frac{1}{k_b} P_2 \right) = \frac{k_a}{\sqrt{2}} \frac{2}{k_a} P_1 = \sqrt{2}\, P_1 \qquad \text{(XX.7.5.1)}$$

(2) Element (b)

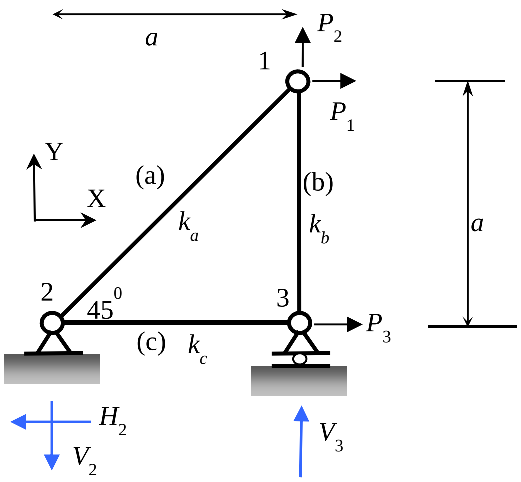
$$N_b = k_b \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_3 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} = k_b D_2 = k_b \left(-\frac{1}{k_b} P_1 + \frac{1}{k_b} P_2 \right) = P_2 - P_1 \qquad \text{(XX.7.5.2)}$$

(3) Element (c)

$$N_c = k_c\left[-1 \quad 0 \quad 1 \quad 0\right]\begin{Bmatrix}0\\0\\D_3\\0\end{Bmatrix} = k_c\ D_3 = k_c \times \frac{P_3}{k_c} = P_3 \qquad \text{(XX.7.5.3)}$$

XX.7.6 Comparison

It disputes about comparison between axial forces of truss members based on static equilibrium equation, and the axial forces calculated from Finite Element analysis.



As shown in above figure, the reactant forces H_2, V_2 and V_3 are calculated based on static equilibrium equation as

$$\begin{aligned} \Sigma X &= 0 &&: H_2 = P_1 + P_3 \\ \Sigma Y &= 0 &&: V_2 + V_3 + P_2 = 0 \\ \Sigma M_{(2)} &= 0 &&: V_3 a + P_2 a = P_1 a \end{aligned} \qquad \text{(XX.7.6.1)}$$

in which the 3rd equation in Eq.(XX.7.6.1) expresses the equilibrium of moment at Node 2. From the result of above equations, we can get the reactant forces H_2, V_2 and V_3.

$$\begin{aligned} V_3 &= P_1 - P_2 \\ V_2 &= -V_3 - P_2 = -P_1 + P_2 - P_2 = -P_1 \\ H_2 &= P_1 + P_3 \end{aligned} \qquad \text{(XX.7.6.2)}$$

From the result of above equations, the axial force of each member is calculated as

$$\begin{aligned} & N_b + V_3 = 0 \ \rightarrow\ N_b = -V_3 = -P_1 + P_2 \\ & N_c = P_3 \\ & N_a \sin 45^o + V_2 = 0 \ \rightarrow\ \frac{1}{\sqrt{2}} N_a - P_1 = 0 \ \rightarrow\ N_a = \sqrt{2}\, P_1 \end{aligned} \qquad \text{(XX.7.6.3)}$$

It is confirmed that the axial force of each truss member based on the static equilibrium equation is in agreement with the axial force calculated by Finite Element analysis.

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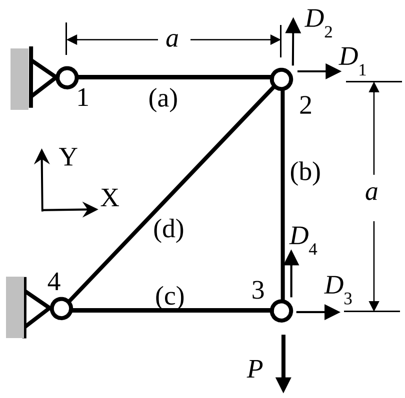
[Example BB.1]
Compute global stiffness matrix of the truss structure shown in Fig.BB.1. There are 4 nodes and 4 elements making up the structure. The young's modulus E and the sectional area A of each truss member are constant. Two dimensional analysis is considered so each node is constrained to move in only the X or Y direction. There are 4 degrees of freedom. Nodal displacements are assumed to be D_1, D_2, D_3 and D_4 as shown in Fig.BB.1. Applied external forces are assumed to be P_1, P_2, P_3 and P_4 corresponding to D_1, D_2, D_3 and D_4, respectively. As loading condition, external load P act downward at Node 3.



BB.1 Truss structure

(1) Stiffness matrix for each element

(a) Element (a) [Node 1 → Node 2]

$$\theta_a = 0 \text{ deg.},\ \sin\theta_a = 0,\ \cos\theta_a = 1,\ \sin\theta_a\cos\theta_a = 0 \quad , \quad k_a = \frac{E_a A_a}{\ell_a} = \frac{EA}{\sqrt{2}\,a}$$

$$\begin{Bmatrix} P_{1a} \\ Q_{1a} \\ P_{2a} \\ Q_{2a} \end{Bmatrix} = k_a \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_{1a} \\ V_{1a} \\ U_{2a} \\ V_{2a} \end{Bmatrix} = k_a \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} \tag{EE.1}$$

$$P_{2a} = k_a D_1 \quad , \quad Q_{2a} = 0 \tag{EE.2}$$

(b) Element (b) [Node 3 → Node 2]

$$\theta_b = 90 \text{ deg.},\ \sin\theta_b = 1,\ \cos\theta_b = 0,\ \sin\theta_b\cos\theta_b = 0 \quad , \quad k_b = \frac{E_b A_b}{\ell_b} = \frac{EA}{a}$$

$$\begin{Bmatrix} P_{3b} \\ Q_{3b} \\ P_{2b} \\ Q_{2b} \end{Bmatrix} = k_b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_{3b} \\ V_{3b} \\ U_{2b} \\ V_{2b} \end{Bmatrix} = k_b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \\ D_1 \\ D_2 \end{Bmatrix} \tag{EE.3}$$

$$P_{3b} = 0,\ Q_{3b} = -k_b D_2 + k_b D_4,\ P_{2b} = 0,\ Q_{2b} = k_b D_2 - k_b D_4 \tag{EE.4}$$

(c) Element (c) [Node 4 → Node 3]

$$\theta_c = 0 \text{ deg.},\quad \sin\theta_c = 0\ ,\ \cos\theta_c = 1\ ,\ \sin\theta_c\cos\theta_c = 0\quad ,\quad k_c = \frac{E_c A_c}{\ell_c} = \frac{E\,A}{a}$$

$$\begin{Bmatrix} P_{4c} \\ Q_{4c} \\ P_{3c} \\ Q_{3c} \end{Bmatrix} = k_c \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_{4c} \\ V_{4c} \\ U_{3c} \\ V_{3c} \end{Bmatrix} = k_c \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ D_3 \\ D_4 \end{Bmatrix} \tag{EE.5}$$

$$P_{3c} = k_c D_3 \quad , \quad Q_{3c} = 0 \tag{EE.6}$$

(d) Element (d)

$$\theta_c = 45 \text{ deg.},\quad \sin\theta_d = \frac{1}{\sqrt{2}}\ ,\ \cos\theta_d = \frac{1}{\sqrt{2}}\ ,\ \sin\theta_d\cos\theta_d = \frac{1}{2}\quad ,\quad k_d = \frac{E_d A_d}{\ell_d} = \frac{E\,A}{\sqrt{2}\ a}$$

$$\begin{Bmatrix} P_{4d} \\ Q_{4d} \\ P_{2d} \\ Q_{2d} \end{Bmatrix} = \frac{k_d}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} U_{4d} \\ V_{4d} \\ U_{2d} \\ V_{2d} \end{Bmatrix} = \frac{k_d}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ D_1 \\ D_2 \end{Bmatrix} \tag{EE.7}$$

$$P_{2d} = \frac{k_d}{2} D_1 + \frac{k_d}{2} D_2\ ,\quad Q_{2d} = \frac{k_d}{2} D_1 + \frac{k_d}{2} D_2 \tag{EE.8}$$

(2) Global stiffness matrices

Since there are 4 displacements (or *DOF*), D_1 through D_4, the matrix is 4x4. Assembling all the terms for elements (a), (b) , (c) and (d), we get the complete matrix equation of the structure. From the relation between extarnal forces and nodal forces, the equiliburium equations of forces are expressed as

$$\begin{aligned} P_1 &= P_{2a} + P_{2b} + P_{2d} \\ P_2 &= Q_{2a} + Q_{2b} + Q_{2d} \\ P_3 &= P_{3b} + P_{3c} \\ P_4 &= Q_{3b} + Q_{3c} \end{aligned} \tag{EE.9}$$

Substituting Eqs.(EE.2), (EE.4), (EE.6) and (EE.8) into above equation, we get

$$\begin{aligned} P_1 &= k_a D_1 + 0 + \frac{k_d}{2} D_1 + \frac{k_d}{2} D_2 = \left(k_a + \frac{k_d}{2} \right) D_1 + \frac{k_d}{2} D_2 \\ P_2 &= 0 + k_b D_2 - k_b D_4 + \frac{k_d}{2} D_1 + \frac{k_d}{2} D_2 = \frac{k_d}{2} D_1 + \left(k_b + \frac{k_d}{2} \right) D_2 - k_b D_4 \\ P_3 &= 0 + k_c D_3 = k_c D_3 \\ P_4 &= -k_b D_2 + k_b D_4 + 0 = -k_b D_2 + k_b D_4 \end{aligned} \tag{EE.10}$$

In a matrix form

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} P_{2a} + P_{2b} + P_{2d} \\ Q_{2a} + Q_{2b} + Q_{2d} \\ P_{3b} + P_{3c} \\ Q_{3b} + Q_{3c} \end{Bmatrix} = \begin{bmatrix} k_a + \dfrac{k_d}{2} & \dfrac{k_d}{2} & 0 & 0 \\ \dfrac{k_d}{2} & k_b + \dfrac{k_d}{2} & 0 & -k_b \\ 0 & 0 & k_c & 0 \\ 0 & -k_b & 0 & k_b \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} \qquad \text{(EE.11)}$$

Considering the conditions of external forces ($P_1 = P_2 = P_3 = 0$ and $P_4 = -P$),

$$\begin{bmatrix} k_a + \dfrac{k_d}{2} & \dfrac{k_d}{2} & 0 & 0 \\ \dfrac{k_d}{2} & k_b + \dfrac{k_d}{2} & 0 & -k_b \\ 0 & 0 & k_c & 0 \\ 0 & -k_b & 0 & k_b \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -P \end{Bmatrix} \qquad \text{(EE.12)}$$